7. GRAPH AND MAP COLOURING

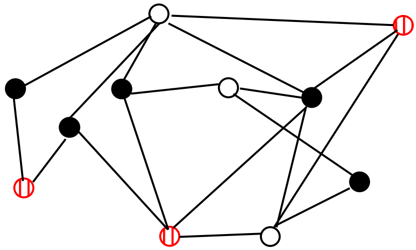
§7.1. Coloured Graphs

We can decorate graphs by colouring either the vertices or the edges, and this raises a number of interesting questions and even leads to some applications.

Suppose we represent the classes at some high school by the vertices of a graph, with vertices being adjacent if there's at least one student who is in both classes. When the graph is coloured, so that adjacent vertices have different colours, these colours can represent different time slots. We want to colour the graph with as few colours as possible.

A ***k*-coloured graph** is a graph where the vertices are coloured with *k*-colours. A ***k*-colourable graph** is a graph where the vertices can be coloured with *k* colours so that adjacent vertices have different colours.

Example 8: Here is a 3-coloured graph.



Since no two adjacent vertices have the same colour this graph is 3-colourable. If the colours represent classes, and adjacent vertices represent classes that must be held at different times, this colouring shows that we can draw up a time-table using only three time-slots.

Note that this graph has no triangles. Perhaps it's 2-colourable. Is it?

§7.2. Graph Colouring

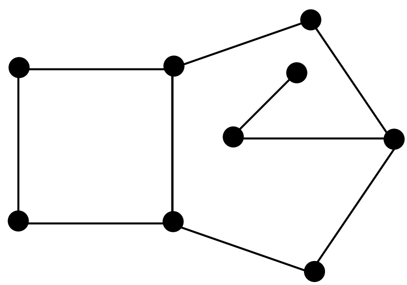
Imagine the task of designing a school timetable. If we ignore the complications of having to find rooms and teachers for the classes we could propose the following very simple model. We suppose that we have a certain number of subjects taken by a certain number of students. We'll suppose that all subjects require just one class. In setting up the timetable we must ensure that if two subjects have at least one student in common we must allocate different times.

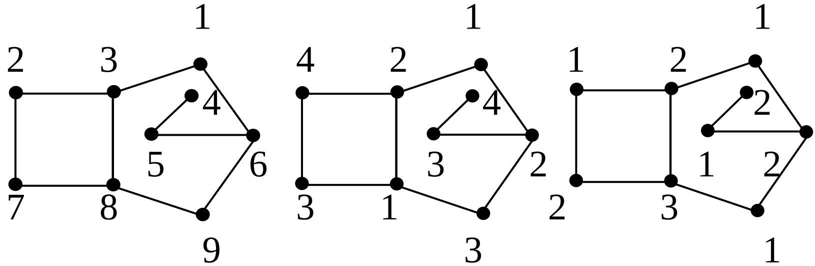
So imagine that the subjects are represented by the vertices of a graph and that we have an edge between subjects that have one or more common students. If we colour the vertices in such a way that adjacent vertices have different colours we can arrange for all subjects that were coloured with the same colour to be held at the same time.

We could simply colour each vertex with a different colour. This would correspond to timetabling each subject at a different time. But there aren't enough hours in the week and it's important to try to use as few time-slots, or as few colours, as possible. This is one application that would benefit from a solution to the problem of colouring graphs with as few colours as possible.

An ***n*-colouring** of a graph is an assignment of 'colours"'1, 2, … , *n* to the vertices of a graph so that adjacent vertices have different colours. The chromatic number, μ(*G*), of a graph *G*, is the *smallest n* for which an *n*-colouring exists.

Example 1: For this graph *G*, μ(G) = 3.

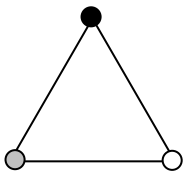




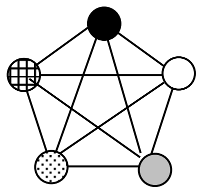
9-colouring 4-colouring 3-colouring

No 2-coloring exists. Around the pentagon we'd have to alternate colours, but because of the odd number of sides we'd end up with two adjacent vertices with the same colour. So 3 is the minimum number of colours that can do the job.

Example 2: The chromatic number of the complete graph K_n is n.
Proof: Since every vertex is adjacent to every other we need to use n distinct colours.



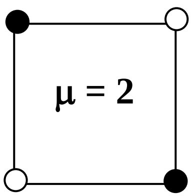
$\mu = 3$

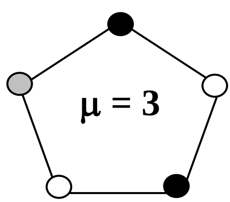


$\mu = 5$

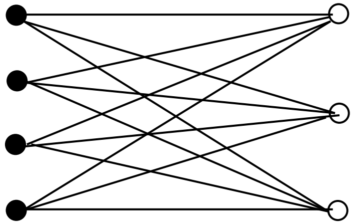
Every graph that contains a subgraph with chromatic number n must itself have chromatic number at least n. So, for example, the chromatic number of any graph that contains a triangle must be at least 3.

Example 3: The chromatic number of an n-sided polygon is: $\begin{cases} 2 \text{ if } n \text{ is even} \\ 3 \text{ if } n \text{ is odd} \end{cases}$



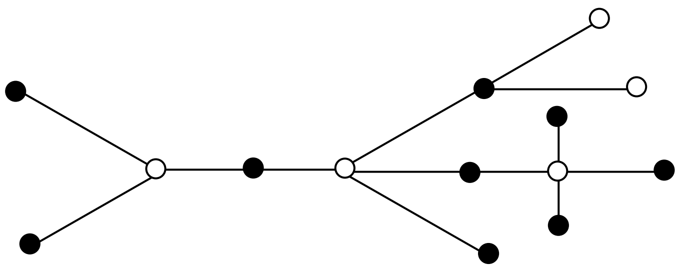


Example 4: The chromatic number of $K_{m,n}$ is 2.
Proof: Colour the *m* vertices one colour and the *n* vertices the other colour.



$\mu(K_{4,3}) = 2$

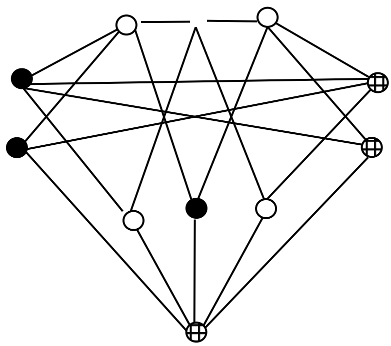
Theorem 1: The chromatic number of a tree (connected graph with no cycles) is 2 (except where there's only one vertex).
Proof: Choose a vertex V_0. Each vertex has a unique distance from V_0. Use one colour to colour those vertices whose distance from V_0 is even and the other colour for those vertices whose distance is odd. Clearly this is a valid 2-colouring.



§7.3. Chromatic Number and Girth

The chromatic number of a graph with a triangle (three mutually adjacent vertices) is clearly at least 3 but a graph without triangles may still have chromatic number 3, as can be seen by example 3. In fact a graph without triangles can have arbitrarily large chromatic number.

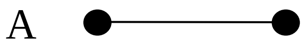
Example 5: The chromatic number of the following graph is 4, yet it contains no triangles.



The top 5 vertices can be coloured with 3 colours and the 5 vertices below them can be coloured with corresponding colours. But then the very bottom vertex is adjacent to all these 3 colours but can be coloured with a 4^{th} colour.

This doesn't prove that a 3-colouring is impossible, but consider the following argument. Suppose the graph could be coloured with 3 colours. Clearly the top 5 vertices definitely require all three colours. For each of these colours there is a vertex with that colour that is adjacent to the other two colours. The corresponding vertex in the next 5 will be adjacent to all those two colours and so must be coloured using the 3rd colour. This means that all 3 colours must be used with the second lot of 5 vertices. But this would mean that the bottom vertex has to be coloured using a 4th colour.

Theorem 2: There are graphs with no triangles of arbitrarily large chromatic number.
Proof: We construct a sequence of graphs as follows.
G_1 has two vertices that are adjacent.

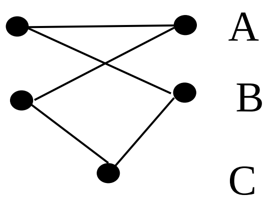


We define G*n*, for all *n*, inductively as follows:
Given G_n, which we'll also call A, we construct a second copy, B, from which we remove all the edges.

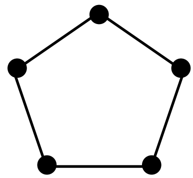
Then, suppose V′ is a vertex in B that corresponds to the vertex V in A. In B we join to *v*′every vertex in A that is adjacent to V′.

Finally we take a graph with one new vertex and no edges, and call this C. We then join that vertex in C to every vertex in B. This new graph we call G_{n+1}.

Here is G_2.



which can be redrawn as a 5-cycle:



This has no triangles and has chromatic number 3.

You will observe that the graph in Example 5 is precisely G_3. It has no triangles and has chromatic number 4. It is easy to prove by induction that G_n has no triangles for all n. Moreover, by the same argument we used to show that the chromatic number of G_3 is 4 we can prove, by induction, that $\mu(G_n) = n + 1$ for all n.

Recall that the girth of a graph is the length of the smallest cycle, we see that G_2 has girth 5 and G_3 has girth 4.

Since G_n is always a subgraph of G_{n+1} it is obvious that the girth of G_n is 4 for all $n \geq 3$. But there do exist graphs with arbitrarily large girth and arbitrarily large chromatic number.

Theorem 3 (ERDOS): For all k there exists a graph with girth > k and chromatic number > k.

Proof: I won't prove this here. However I'd like to point out the interesting fact that this proof of existence is non-constructive. Unlike the previous proof, where we explicitly described a suitable graph, this proof does not. It's a probabilistic proof and is a good example of probabilistic methods.

It is important to realise that the proof doesn't merely show that it is *probably* true. It is a full proof. The proof shows that the probability that a random graph on n vertices has chromatic number at most k is less than 1/3 and that the probability that a random graph has chromatic at most k is also less than 1/3 for sufficiently large n and suitable probability p. This means that the probability of such a random graph satisfying the theorem is at least 1/3. Since there are only finitely many graphs on n vertices this means that there will definitely be such a graph. But to find an explicit graph, even for relatively small values of k, is extremely difficult.

§7.4. Chromatic Polynomials

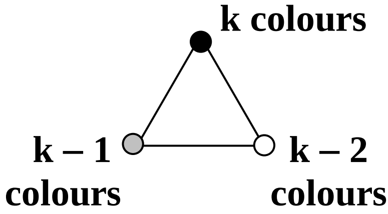
The **chromatic polynomial** of a graph G is the number of ways of colouring G with k colours (a polynomial in k). It is denoted by $\boldsymbol{\Gamma(G)(k)}$.

The **chromatic number** of G is the smallest value of k such that $\Gamma(G)(k)$ is positive.
For simplicity, unless we need to specify the parameter k, we'll write $\Gamma(G)(k)$ simply as $\Gamma(G)$.

Theorem 4: $\Gamma(\mathbf{K}_n) = k(k-1) \ldots (k-n+1)$.
Proof: Order the vertices in some way. There are k colours that can be assigned to the first vertex. Any one

of the remaining colours can be assigned to the second vertex, and so on.

Example 6: $\Gamma(K_3) = k(k-1)(k-2)$.



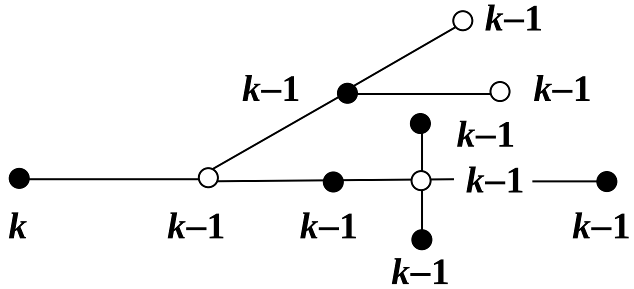
A **tree** is a connected graph in which there are no cycles. In a tree there's a unique path (sequence of edges) from any vertex to any other and so there's a unique distance (counted in terms of the number of edges) between any two vertices. Any vertex whose distance from a given one is greatest must clearly have degree 1. So trees will always have vertices of degree 1.

Theorem 5: If G is a tree with n vertices then

$$\Gamma(G) = k(k-1)^{n-1}.$$

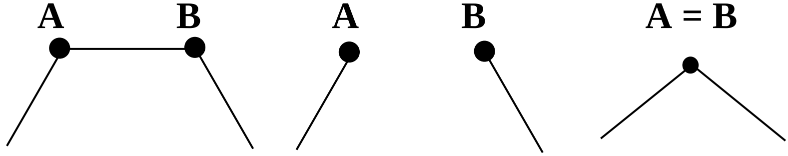
Proof: We use induction on the number of vertices. It's clearly true for $n = 1$.
Suppose the result is true for trees with n vertices and that we have one with $n + 1$ vertices.
Choose any vertex of degree 1 and remove it, together with the associated edge.
By induction the resulting tree can be coloured with k colours in $k(k-1)^{n-1}$ ways.

Reinstating the deleted vertex, it's adjacent to only one coloured vertex and so can be coloured in $k - 1$ ways, giving a total number of colourings of $k(k-1)^n$.

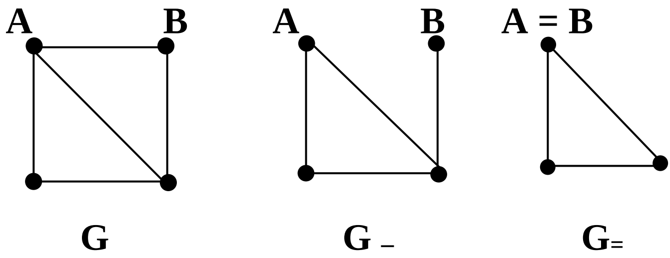


We now show how to compute the chromatic polynomial of a graph. The method is inductive. From a given graph G we produce two simpler graphs G_- and $G_=$ and we use their chromatic polynomials to obtain the chromatic polynomial of G.

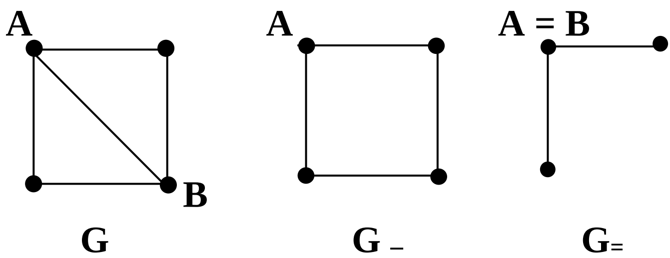
Let G be a graph. Select any two adjacent vertices A and B.
Let $\mathbf{G}_-$ be the same graph with the edge AB deleted.
Let $\mathbf{G}_=$ denote the graph G with vertices A and B identified. (This means they become a single vertex and any edge having either A or B as an endpoint now has, instead, this combined vertex.)



Example 7:



Or choosing a different edge:

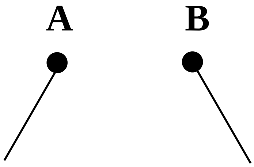
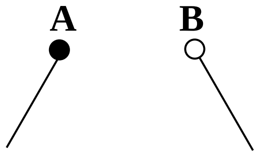
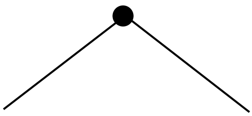
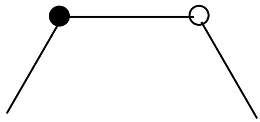


Theorem 6: Let G be a graph. Select any two adjacent vertices A and B.
Then

$$\boldsymbol{\Gamma(G) = \Gamma(G_{-}) - \Gamma(G_{=})}.$$

Proof: If we *k*-colour G_{-} we're free to colour A and B the same colour. But we don't have to. We could just as validly colour them different colours (provided this was consistent with the colourings at the other vertices).

There are two types of colourings of G_{-} in terms of what happens to A and B

A, B are given the SAME colour	**A, B are given DIFFERENT colours**
 **G_-**	 **G_-**
Each of these colourings of G_- gives a valid colouring of $G_=$.  **$G_=$**	Each of these colourings of G_- gives a valid colouring of G.  **G**

Hence $\Gamma_{G_-}(k) = \Gamma_{G_=}(k) + \Gamma_G(k)$. The result now follows algebraically.

Example 8:

$\Gamma($  $) = \Gamma($  $) - \Gamma($  $)$

$= k(k-1)^3 - k(k-1)(k-2)$

$= k(k-1)[(k-1)^2 - (k-2)]$

$= k(k-1)(k^2 - 3k + 3).$

Example 9: $\Gamma($  $) = \Gamma($  $) - \Gamma($  $)$

Now $\Gamma($  $) = \Gamma($  $).(k-1) = k(k-1)^2(k-2)$

since once the triangle has been coloured the remaining vertex can be coloured any colour, except the colour of the vertex to which it is attached.

Also $\Gamma($$) = \Gamma($$) - \Gamma($$)$

$= k(k-1)^4 - k(k-1)(k^2 - 3k + 3)$ from Example 7.

Hence $\Gamma($$)$

$$= k(k-1)^4 - k(k-1)(k^2 - 3k + 3) - k(k-1)^2(k-2)$$
$$= k(k-1)[(k-1)^3 - (k^2 - 3k + 3) - (k-1)(k-2)]$$
$$= k(k-1)[k^3 - 3k^2 + 3k - 1 - k^2 + 3k - 3 - k^2 + 3k - 2]$$
$$= k(k-1)(k^3 - 5k^2 + 9k - 6)$$
$$= k(k-1)(k-2)(k^2 - 3k + 3).$$

It follows that the chromatic number of  is 3. This was pretty obvious without all that calculation, but for a much more complicated graph, where the chromatic number is not so obvious, this could be a useful technique. Also, it could form the basis for a computer program to compute the chromatic number of a graph.

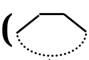
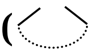
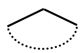
Theorem 7: If $\mathbf{C}_n$ is a cycle with n vertices then

$$\Gamma(\mathbf{C}_n) = (k-1)^n + (-1)^n(k-1).$$

Proof: Induction on n.

$\Gamma(\mathbf{C}_3) = \Gamma(\mathbf{K}_3) = k(k-1)(k-2)$ so it holds for $n = 3$.

Suppose that it holds for n and let $G = \mathbf{C}_{n+1}$.

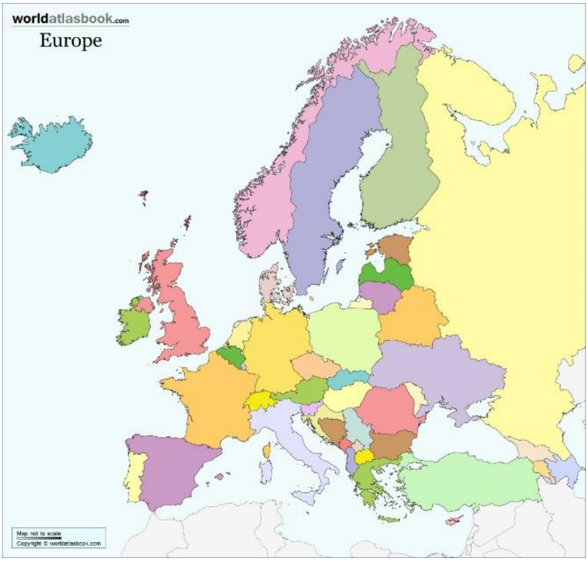
$\boldsymbol{\Gamma($$) = \Gamma($$) - \Gamma($$)}$.

$$\Gamma(\mathbf{C}_{n+1}) = k(k-1)^n - \Gamma(\mathbf{C}_n)$$
$$= k(k-1)^n - (k-1)^n - (-1)^n(k-1)$$
$$= (k-1)^{n+1} + (-1)^{n+1}(k-1).$$

Hence it holds for $n + 1$ and so, by induction, for all n.

§7.5. Map Colouring

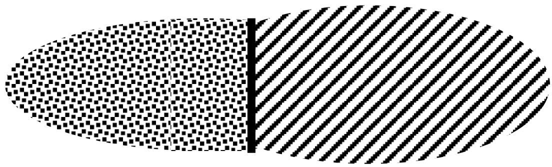
Graph colouring is a purely combinatorial exercise that doesn't involve any topology. However it's a useful lead-in to map colouring. As with graph colouring, when colouring maps, we must colour adjacent objects with different colours.



But in the case of map colouring the objects are the regions, or faces, in the map and 'adjacent' means that they share a common boundary.

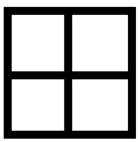
Adjacent counties in a map of the world are coloured differently to make the borders clear. Though there's no need to use the minimum number of colours it's an interesting question to ask "what is the minimum number of colours we can use?" And we don't just stick to real maps, of countries. We consider all possible maps that can be drawn.

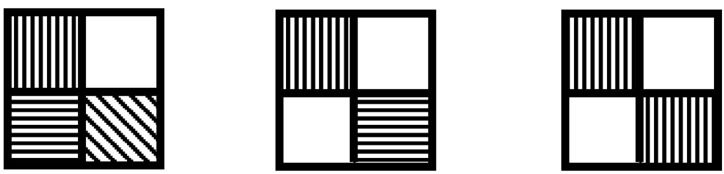
An ***n*-colouring** of a map on a surface is a way of assigning one of *n* labels (colours) to each face so that adjacent faces have different colours.



A map is ***n*-colourable** if an n-colouring of it exists. The **chromatic number**, μ(M), of a map, M, is the *smallest n* for which *M* is *n*-colourable.

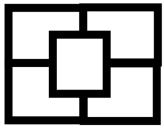
Example 10: The chromatic number of the following map is 2:

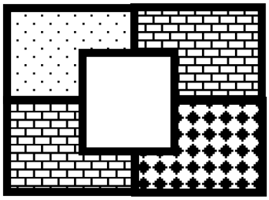




4-colouring 3-colouring 2-colouring
but no 1-colouring exists

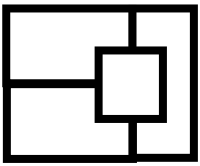
Example 11: The chromatic number of the following map is 3:

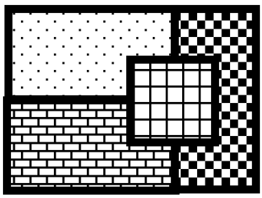




3-colouring
but no 2-colouring exists

Example 12: The chromatic number of the following map is 4:





4-colouring
but no 3-coloring exists

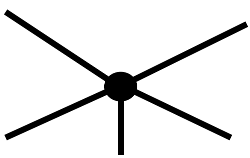
The **chromatic number**, μ(S), of a surface, S, is the *largest* chromatic number for any map that can be drawn on it. Now there's no *a priori* reason why this couldn't be infinite because we can draw larger and larger maps on the surface, requiring more and more colours. However we'll show that every surface has a finite chromatic number. This means that there's an upper bound to the chromatic numbers of maps on a given surface.

Upper bounds are harder to arrive at than lower bounds. To find a lower bound for the chromatic number of a surface we only have to draw a map on the surface, and the chromatic number of the surface will have to be at least as big as the chromatic number of that map. For example, the fact that in example 9 we have a map on a disk with chromatic number 2 means that $\mu(\text{disk}) \geq 2$.

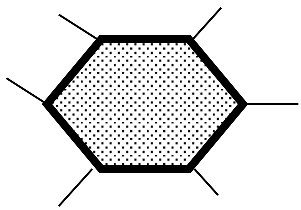
Example 13: $\mu(\text{disk}) \geq 4$.
In example 11 we had a map, M, on a disk with $\mu(M) = 4$. Hence $\mu(\text{disk}) \geq 4$. If there existed another map on the disk that needs 5 colours, we'd have $\mu(\text{disk}) \geq 5$. But in fact it can be shown that every map on a disk can be coloured validly with 4 colours so in fact $\mu(\text{disk}) = 4$. This is the famous Four Colour Theorem, proved in 1976.

Recall that the **degree of a vertex** is the number of edges at that vertex. With a map we define the **degree of a face** to be the number of edges surrounding that face.

Example 14:



a vertex of degree 5



a face of degree 6

Clearly holes in a surface don't affect its chromatic number. So the chromatic number of a disk (sphere with one hole) will be the same as that of a sphere, or a cylinder (sphere with two holes). If required to find the chromatic number of a surface therefore, the first step is to remove all holes – that is **replace the surface by the corresponding surface with no holes**.

This is exactly what we do for embedding questions. Also, like embedding, we **remove vertices of degrees 1 and 2** from a map. When removing a vertex of degree 1 we also remove the corresponding edge. When removing a vertex of degree 2 we combine the two edges into a single edge. In what follows we assume that this has already been done and that
the surfaces have no boundaries and the degrees of the vertices are at least 3.

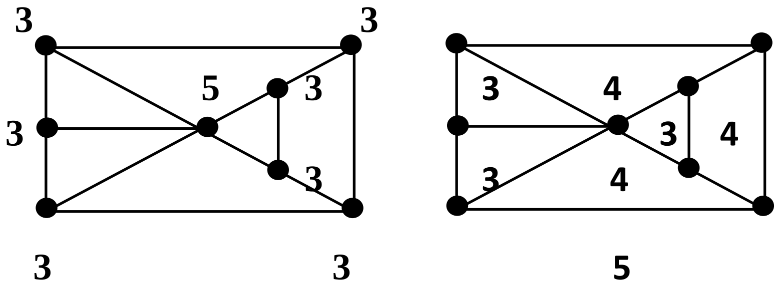
For a map*M* we define:

V = the number of vertices;
E = the number of edges;
F = the number of faces;
***v* = the average degree of the vertices and**
***f* = the average degree of the faces.**

Then $\boldsymbol{v} = \frac{\mathbf{2E}}{\mathbf{V}}$. Why not $\frac{E}{V}$? The answer is that each edge contributes to the degree of each of its two endpoints. If we chopped each edge in two, so that the half edges were associated with only one vertex, then we'd have $2E$ of these half edges.

Similarly, $\boldsymbol{f} = \dfrac{\mathbf{2E}}{\mathbf{F}}$. Again, each edge is shared by two faces.

Example 15:



Here we've labelled the vertices and faces with their degrees. Note that the outside has to be considered too, because we're regarding this as a map on a sphere. The average degree of the vertices is thus:

$$\frac{3+3+3+3+3+5+3+3}{8} = \frac{26}{8}.$$

Note that E = 13, so this is simply $\dfrac{2E}{V}$.

The average degree of the faces is:

$$\frac{3+4+3+4+3+4+5}{7} = \frac{26}{7}.$$

Note that this is simply $\dfrac{2E}{F}$.

Theorem 8: Suppose that M is a map (with all vertices of degree ≥ 3) on a surface S (with no holes), with Euler characteristic χ. Let V, F and E be the numbers of vertices, faces and edges for M and let v be the average degree of the vertices and let f be the average degree of the faces. Then:

$$(f-2)(v-2) = 4 - \frac{2f\chi}{V} = 4 - \frac{2v\chi}{F}.$$

Proof: In counting the total number of edges by adding the degrees of the vertices or faces, each edge gets counted twice (joins 2 vertices) and each face gets counted twice (borders 2 faces).

Hence $f\,F = vV = 2E$.

Thus $\frac{f}{V} = \frac{v}{F}$, establishing the second equality of the theorem.

Now $v = \frac{2E}{V}$

$$= \frac{2(V + F - \chi)}{V}$$

$$= 2 + \frac{2F}{V} - \frac{2\chi}{V}$$

$$= 2 + \frac{2v}{f} - \frac{2\chi}{V}$$

Thus $vf = 2f + 2v - \frac{2f\chi}{V}$ and so

$$(f-2)(v-2) = vf - 2f - 2v + 4 = 4 - \frac{2f\chi}{V}.$$

§7.6. Polyhedra

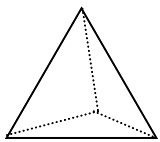
A **polyhedron** is a region in 3-dimensional space, homeomorphic to a sphere and bounded by flat sides each of which is a polygon. There's no topological reason why the faces have to be flat, or the edges straight so it's convenient to consider polyhedra as maps on a sphere. Theorem 1 can therefore be applied to polyhedra by taking $\chi = 2$.

Theorem 9: The average number of edges per face for any polyhedron is less than 6.

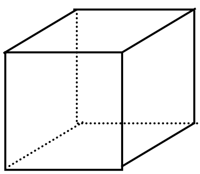
Proof: We use the usual symbols for maps.

Since $v \geq 3$ we have $f - 2 \leq (f - 2)(v - 2) = 4 - \frac{4f}{V} < 4$, giving $f < 6$.

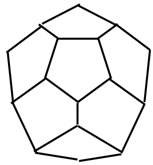
A consequence of this theorem is that there's no polyhedron where every face is a hexagon. There are, however, polyhedra where every face is a triangle ($f = 3$)



or a square ($f = 4$)



or even a pentagon ($f = 5$).



But that's as far as we can go.

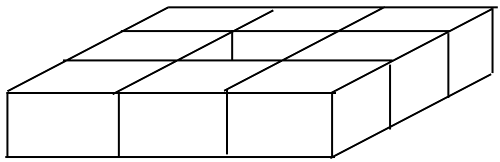
If we extend the idea of a polyhedron to maps on other surfaces we can make the following conclusion about toroidal polyhedra (those which are homeomorphic to a torus).

Theorem 10: For a map on a torus we have:

$$(f - 2)(v - 2) = 4.$$

Proof: This follows from theorem 1 by putting $\chi = 0$.

Example 16: The following is a 3×3 array of cubes, with the centre one removed. It can be considered as a map on a torus.



There are eight vertices of degree 3, 16 of degree 4 and eight of degree 5. The average degree of the vertices is therefore $v = 4$. Since each face is a square, the average degree of the faces is $f = 4$.

A consequence of theorem 6 is that for a regular polyhedron on a torus the faces must be hexagons, squares or triangles (since $v - 2$ must be an integer and so $f - 2$ must divide 4).

§7.7. Heawood's Theorem

Heawood's Theorem gives an upper bound on the chromatic number of any surface in terms of its Euler characteristic. It assumes that there are no holes (these must be patched up) but it works for both orientable and non-orientable surfaces.

Theorem 11: For a map with F faces on a surface of weight χ_0 the average face degree is at most

$$6 + \frac{6(\chi_0 - 2)}{F}.$$

Proof: $(f - 2)(v - 2) = 4 - \dfrac{2v\chi}{F}$.

Hence $f - 2 = \dfrac{4}{v-2} - \dfrac{2\chi}{F}\left(\dfrac{v}{v-2}\right)$

$$= \frac{4}{v-2} - \frac{2\chi}{F}\left(1 + \frac{2}{v-2}\right)$$

Thus $f = 2\left(1 + \dfrac{2}{v-2}\right)\left(1 - \dfrac{\chi}{F}\right)$

$$\leq 6\left(1 - \frac{\chi}{F}\right), \text{ because } v \geq 3.$$

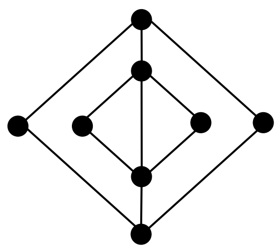
So $f \leq 6 + \dfrac{6(\chi_0 - 2)}{F}$.

Example:

V = 8

E = 11

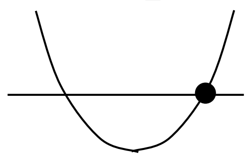
F = 5 (including the outside)



$\chi = 8 + 5 - 11 = 2$ (sphere)
$\chi_0 = 0$
$$f = \frac{6+6+3+3+4}{5} = \frac{22}{5}$$
$$v = \frac{3+2+3+2+3+3+2+2}{8} = \frac{20}{8} = \frac{5}{2}$$

A **regular map** is one where no face has a border with itself and any two different faces have at most one edge separating them.

Theorem 12: For a regular map on a surface of weight $\chi_0 \geq 2$ we have:
$$f \leq \frac{5 + \sqrt{24\chi_0 + 1}}{2}.$$
Proof: Let F be the number of faces.
Since $f \leq F - 1$ (a given face in a regular map does not border more than all the remaining faces) we have:
$f \leq 6 + \dfrac{6(\chi_0 - 2)}{f+1}$ so $f^2 - 5f + 6(1 - \chi_0) \leq 0$. The greater of the two zeros of this quadratic is $\dfrac{5 + \sqrt{24\chi_0 + 1}}{2}$.



Theorem 13: (HEAWOOD 1910) Let S be a surface with no boundaries and weight χ_0.

$$\text{If } \chi_0 \geq 2 \text{ then } \mu(S) \leq \text{INT}\left[\frac{7+\sqrt{24\chi_0+1}}{2}\right].$$

Also μ(sphere) ≤ 6 and μ(projective plane) ≤ 6.

Proof: Suppose $\chi_0 \geq 2$ and let $k = \dfrac{7+\sqrt{24\chi_0+1}}{2}$ and let n be the integer part of k.

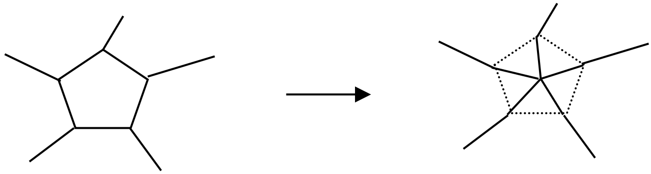
We prove by induction on the number of faces that every regular map on S can be n-coloured.

A map with one face can be coloured with a single colour and since $n \geq 4$ we can easily do it with n colours.

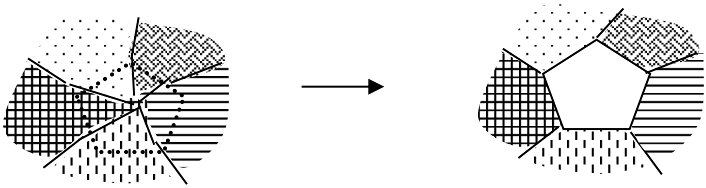
Now consider the general case. Suppose we have a certain map with at least two faces and assume that any map with fewer faces can be n-coloured.

Since $f \leq k-1$ there exists a face whose degree is at most $k-1$ and its degree is at most $n-1$.

Shrink that face to a point. All edges having one of the vertices of the chosen face as an endpoint will now meet at the same vertex.



By induction we can *n*-colour this new map. Now replace the face in the now coloured map. It remains to show that this face can be validly coloured.



Since it borders at most $n-1$ other faces, at most $n-1$ colours must be avoided if we are to colour this remaining face. With *n* colours available this can be done.

If $\chi_0 = 1$ or 2 we can repeat the above argument by taking $n = 6$.

Heawood only gave an upper bound. But surprisingly there are only two surfaces for which this upper bound is bigger than it need be. For all other surfaces Heawood's upper bound can be shown to be the actual value. The exceptions are the Klein bottle and the sphere.

Heawood says $\mu(KB) \leq 7$ but in fact $\mu(KB) = 6$.

Heawood says $\mu(\text{sphere}) \leq 6$ while in fact $\mu(\text{sphere}) = 4$.

If S is a surface with no boundaries and weight χ_0:

$$\mu(S) = \begin{cases} \text{INT}\left[\dfrac{7 + \sqrt{24\chi_0 + 1}}{2}\right] \text{ except that} \\ \mu(\text{sphere}) = 4 \text{ and} \\ \mu(\text{KB}) = 6 \end{cases}.$$

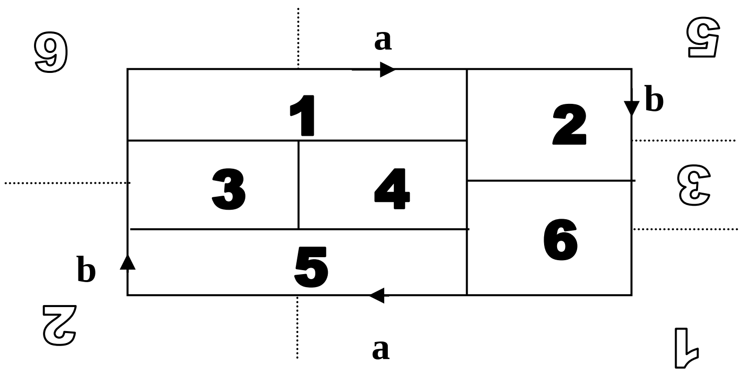
In many cases we'll verify the chromatic number by using Heawood to get an upper bound and by constructing a suitable map to get the same value as a lower bound.

Example 17: μ(projective plane) = 6.

Proof: Heawood says that $\mu \le \text{INT}\left[\dfrac{7 + \sqrt{24\chi_0 + 1}}{2}\right]$

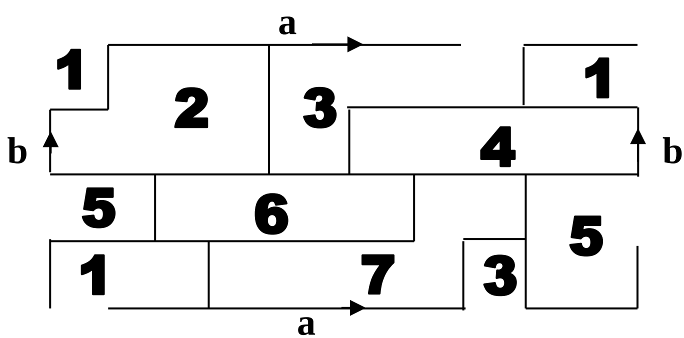
$$\text{So } \mu \le \text{INT}\left[\dfrac{7 + \sqrt{25}}{2}\right] = 6.$$

The following map on the projective plane requires 6 colours so $\mu \ge 6$. Hence $\mu = 6$.



Note that, because of the orientation of the edges, if we cross over the edge ***a*** from region 6 we enter region 1, but it appears upside down, relative to what it would appear if we reached region 1 by travelling through region 4.

Example 18: μ(torus) = 7.
Proof: Heawood says $\mu \leq 7$. The following map on the torus requires 7 colours and so $\mu = 7$.



Each face is adjacent to every other face.

It's remarkable that we can readily establish the chromatic number of surfaces like the projective plane and the torus while something so simple as the sphere (or equivalently the disk) is much more elusive.

We'll soon prove that an upper bound for μ(sphere) is 5. But the proof that it is in fact 4 is exceedingly complicated.

§7.8. The 4-Colour Theorem

The Four Colour Theorem is one of the most celebrated theorems of all time. It began as a question in 1852 when Francis Guthrie, who was drawing and colouring a map of the counties of England, wondered whether four colours are enough. Guthrie had studied under the mathematician Augustus de Morgan at University College in London and his brother, Frederick was then studying mathematics under de Morgan. So Francis passed on the question to de Morgan through his brother.

De Morgan wrote: “A student of mine asked me today to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured – four colours may be wanted but not more – the following is his case in which four colours are wanted. Query: cannot a necessity for five or more be invented?”

Over the next few decades it became the Four Colour Conjecture. Many ‘proofs’ were published and several of them stood for a number of years before they were shown to be wrong. It took over 100 years before, in 1976, it was finally proved by Appel and Haken.

But the proof caused a lot of controversy in that it was the first theorem in history that was proved by a computer program. Of course no computer program could consider the infinitely many possible maps. What Appel and Haken did was to use standard mathematical reasoning to reduce this to 1,834 maps. If all these could be 4-coloured then every map could be 4-coloured.

The computer program then, laboriously, considered each of these maps and, indeed, showed that every one of them was 4-colourable.

But many mathematicians didn't trust a computer program. However over the years the number of cases was reduced, and more efficient programs for dealing with them were written. To this day no proof exists that completely avoids the use of a computer, but it is generally accepted by the mathematical community that the Four Colour Conjecture is now the Four Colour Theorem.

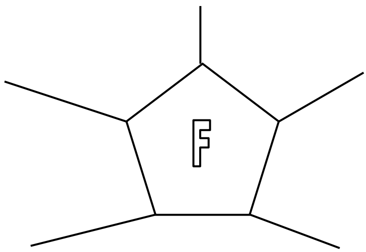
Appel and Haken were at the University of Illinois when they published their proof and the local postal authorities were so proud of this discovery that for many years they franked letters that passed through their hands with the words FOUR COLORS SUFFICE. Indeed they were still using this slogan in 1994 as this picture shows.

§7.9. The 5-Colour Theorem

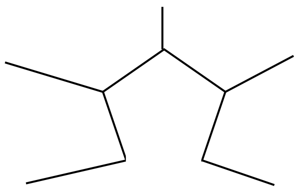
Theorem 14: μ(sphere) ≤ 5. In other words, every map on a sphere can be coloured using 5 colours.

Proof: Suppose there's a map M on the sphere that isn't 5-colourable. We may suppose that we have a minimal counterexample that is, a map where every map with fewer faces is 5-colourable.

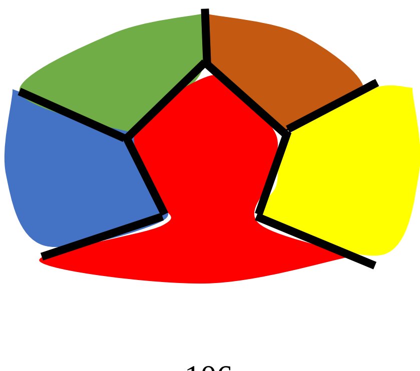
Let f be the average number of edges per face. Since $f < 6$, M has a face F with at most 5 edges.



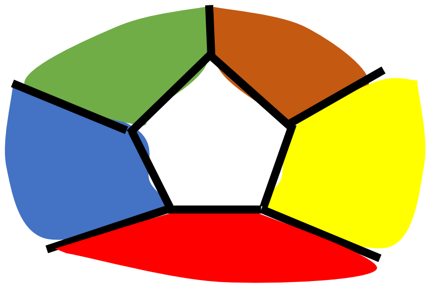
Remove one of these edges and combine the two faces on either side into a single face.



This map has fewer faces than M and so is 5-colourable.



Having 5-coloured this slightly smaller map, now replace the edge and leave F uncoloured.

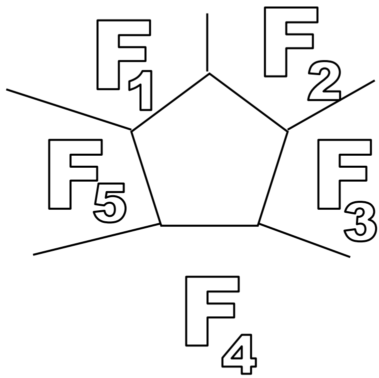


If there are less than 5 different colours around F then there's one available with which to colour F. But this contradicts our assumption that M is not 5-colourable.

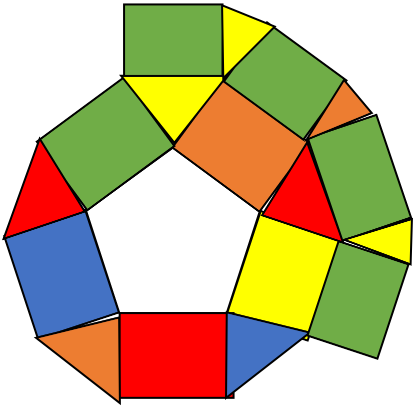
Hence F must be adjacent to exactly 5 faces and these must be coloured using 5 distinct colours. Since there's no colour left that we can use for F we'll need to modify the existing colouring.

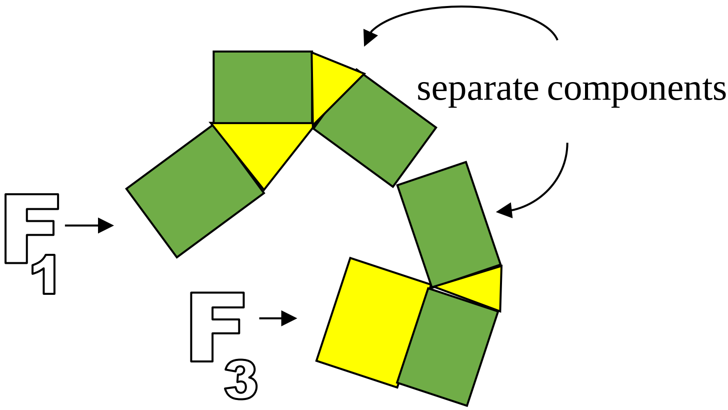
Let the faces surrounding F be F_1, F_2, F_3, F_4 and F_5 (in order around the outside of F, either clockwise or anticlockwise).

Let the corresponding colours be C_1, C_2, C_3, C_4 and C_5.



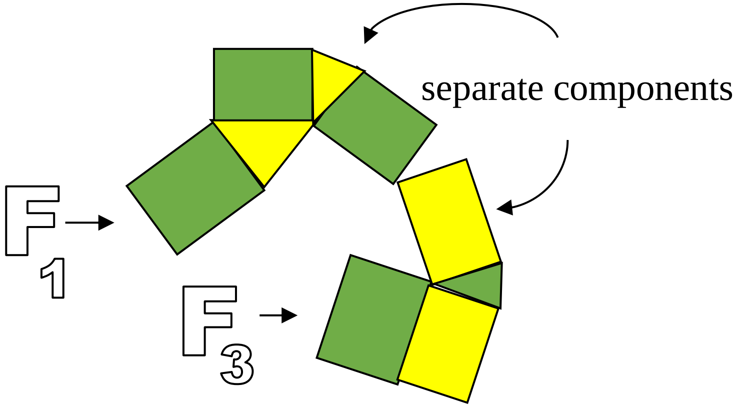
Delete all faces except those coloured C_1 (green) or C_3 (yellow). This may well disconnect M into several components.



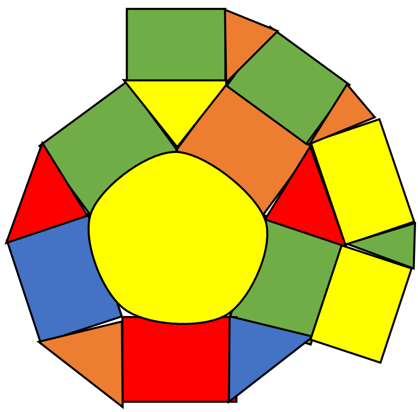


If F_1 and F_3 are in different components of this new map (in other words if you can't go from F_1 to F_3 by sticking to faces coloured C_1 and C_3) we could swap colours C_1 and C_3 in one of these components so that now

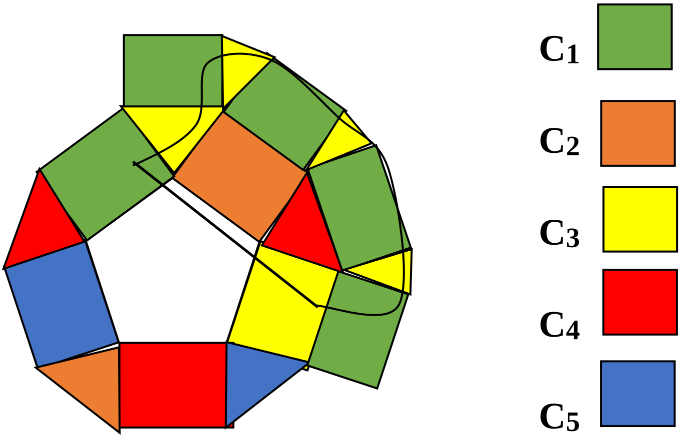
F_1 and F_3 have the same colour (and, of course, maintaining a valid colouring).



But now F would be surrounded by only 4 colours and so it could be validly coloured, to give a 5-colouring of M.



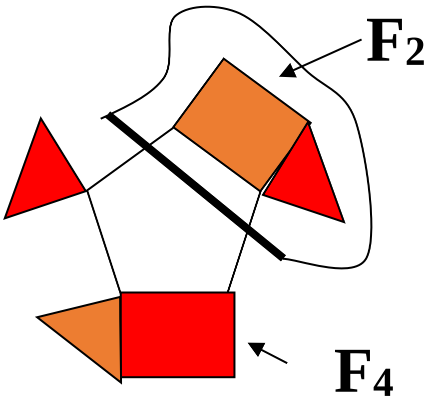
Since our assumption is that M doesn't have a 5-colouring we conclude that F_1 and F_3 must be in the same component. That is, we can only get from one to the other by passing through C_1 (green) or C_3 (yellow) coloured faces. That is, apart from the path that goes directly across the so-far uncoloured face F.



You'll notice that the brown triangle in the top-right of the original diagram has been changed to yellow. We proved that F_1 and F_3 must be in the same component, that is one can get from one to the other by only passing through green and yellow faces. This change is just one possibility of doing this to illustrate the general argument. Of course there may be a much longer path using faces that are not shown.

The direct path, and the one that passes through the blue and green coloured faces, together make up a closed path. Such a closed path must separate F_2 and F_4. That is, one of F_2 and F_4 is on the inside of this closed path and the other is on the outside. (Of course, being on a sphere we can't say unambiguously which is 'inside' and which is 'outside', but that doesn't matter.) In the example it is F_2 that is inside

This means that if we were to delete from M all faces except those coloured C_2 (brown) and C_4 (red) the faces F_2 and F_4 must be in separate components.



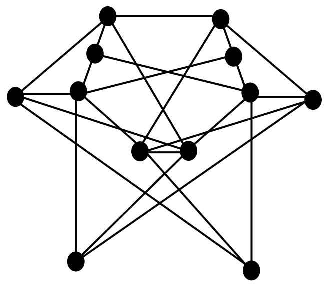
So we could swap colours C_2 and C_4 in one of these components and not in the other so that F_2 and F_4 now have the same colour. As before, this frees up one of the 5 colours with which to colour F and so complete the 5-colouring of M. Again this is a contradiction.

So we can't escape a contradiction. It follows that our assumption that there exists a non 5-colourable map on the sphere must be false. Hence every map on the sphere can be 5-coloured.

EXERCISES FOR CHAPTER 6

Exercise 1:

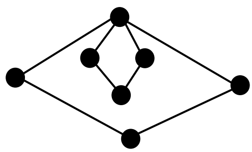
Find the chromatic number of the following graph:



Exercise 2:

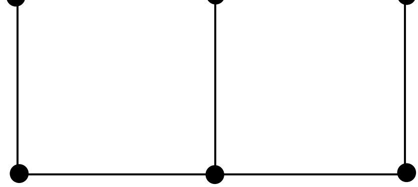
(i) Find the chromatic polynomial of the following graph and express it in factorised form.

(ii) Use this polynomial to find the number of ways of colouring the graph in (i) with 2 colours? How many ways with 3 colours?



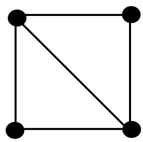
Exercise 3:

Find the number of ways of colouring the vertices of the following graph with 3 colours so that adjacent vertices have different colours.



Exercise 4: Find the number of ways of colouring the vertices of a square with k colours so that adjacent vertices have different colours.

Exercise 5:
Find the number of ways of colouring the vertices of this graph, with 6 colours available, so that adjacent vertices have different colours.



Exercise 6:
(i) Draw a diagram for $\mathbf{K}_6$, the complete graph on 6 vertices. Draw a square with identified edges that represents the Klein bottle.

(ii) Find a 2-cell embedding of $\mathbf{K}_6$ in the Klein bottle.

(iii) Use this to draw, on a separate diagram, a map on the Klein bottle with six regions, each adjacent to the other five. (Finish with a clear picture in which only the edges of the map, and not the original graph, are visible. Label the regions 1 to 6.

(iv) Let μ be chromatic number of the Klein bottle. What does your answer to (iv) tell you about μ?

(v) Heawood's formula gives an upper bound for μ. Calculate this upper bound.

(vi) Combine your answers to (v) and (vi) to determine μ as closely as possible. (Only use what you have proved. Do not simply quote the known value of μ.)

Exercise 7:
Let S be the sum of 3 projective planes and let n be the chromatic number of S.

(a) Find n.

(b) Construct a polygon with identified edges, P, to represent S.

(c) Draw an embedding of $\mathbf{K}_n$, the complete graph on n vertices, on S.

(d) Hence, or otherwise draw a map on P that requires n colours.

Exercise 8:
Find the chromatic numbers of the following surfaces:

(a) the Möbius Band;
(b) a cylinder with 3 holes;
(c) a sum of 10 projective planes.

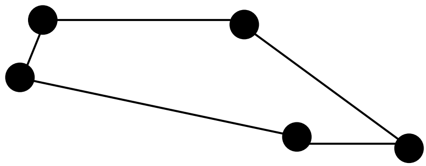
[You may assume that Heawood's upper bound is the actual value for all surfaces with no boundaries except the sphere and the Klein Bottle.]

Exercise 9: (a) A regulation soccer ball is made up of patches, each of which is either a regular pentagon or a regular hexagon. Each vertex is surrounded by three patches. The pentagons are coloured black and the hexagons are coloured white. Based on this information alone show that there must be 12 pentagons.

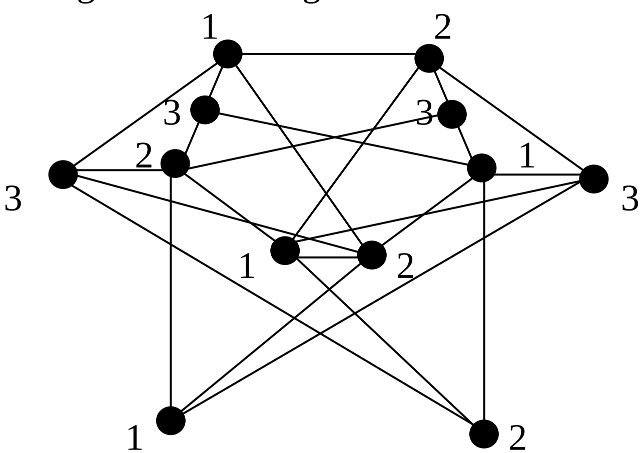
(b) Consider the pattern on a soccer ball to be a map on a sphere. It is made up of 12 pentagons, as predicted by (a), and 20 hexagons. Find the number of vertices and edges of this graph.

SOLUTIONS FOR CHAPTER 6

Exercise 1:
Since the graph contains a cycle of length 5 there is clearly no 2-colouring.

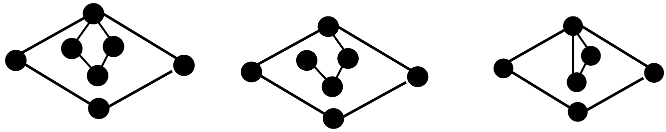


The following is a 3-coluring.



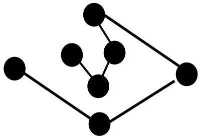
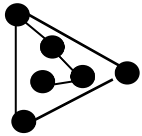
The chromatic number of this graph is therefore **3**.

Exercise 2:
(i) Consider the following graphs:

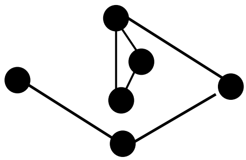
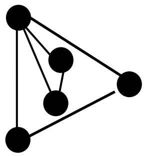


G_1 G_2 G_3

Then $\Gamma(G_1) = \Gamma(G_2) - \Gamma(G_3)$.

Let G_4 =  and G_5 = 

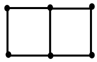
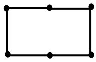
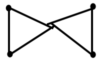
Then $\Gamma(G_2) = \Gamma(G_4) - \Gamma(G_5)$.
Clearly $\Gamma(G_4) = k(k-1)^6$ and $\Gamma(G_5)$
$= k(k-1)^4(k-2)$.
Hence $\Gamma(G_2) = k(k-1)^6 - k(k-1)^4(k-2)$
$= k(k-1)^4(k^2 - 3k + 3)$.

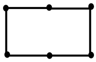
Let G_6 =  and G_7 = 

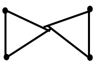
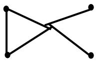
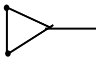
Then $\Gamma(G_3) = \Gamma(G_6) - \Gamma(G_7)$.
Clearly $\Gamma(G_6) = k(k-1)^4(k-2)$ and
$\Gamma(G_7) = k(k-1)^2(k-2)^2$.
Hence $\Gamma(G_3) = k(k-1)^4(k-2) - \mathrm{k}(\mathrm{k}-1)^2(\mathrm{k}-2)^2$
$= k(k-1)^2(k-2)(k^2 - 3k + 3)$.
So $\Gamma(G_1)$
$= k(k-1)^4(k^2 - 3k + 3) - k(k-1)^2(k-2)(k^2 - 3k + 3)$
$= \mathbf{k(k-1)^2(k^2 - 3k + 3)^2}$.

(ii) There are **2** ways of colouring the graph with 2 colours and **108** ways of colouring it with 3 colours.

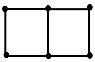
Exercise 3:

Γ () = Γ() – Γ()

Γ() $= (k-1)[(k-1)^5 + 1] = 2^6 + 2$
$= 66$ when $k = 3$.

Γ() = Γ() – Γ()

$= k(k-1)(k-2)(k-1)^2 - k(k-1)(k-2)(k-1)$
$= k(k-1)^2(k-2)^2 = 12$ when $k = 3$.

Hence Γ() = $66 - 12 = 54$ when $k = 3$.

Exercise 4:
$\Gamma = k(k-1)^3 - k(k-1)(k-2)$
$= k(k-1)(k^2 - 2k + 1 - k + 2) = k(k-1)(k^2 - 3k + 3)$.

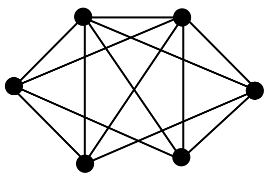
Exercise 5: Let the graph be called G and consider the diagonal.

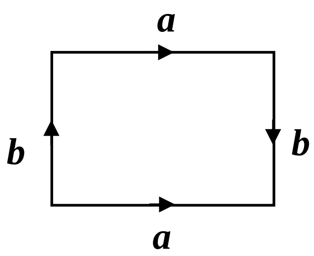
G_{-} is a square and its chromatic polynomial is
$(k-1)^4 + (k-1)$.

$G_{=}$ is a linear graph with 3 vertices and 2 edges.
Its chromatic polynomial is $k(k-1)^2$.
Hence $\mu(G) = (k-1)^4 + (k-1) + k(k-1)^2$.
When $k = 6$, $\mu = 780$.

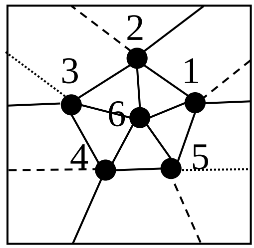
Exercise 6:

(i)





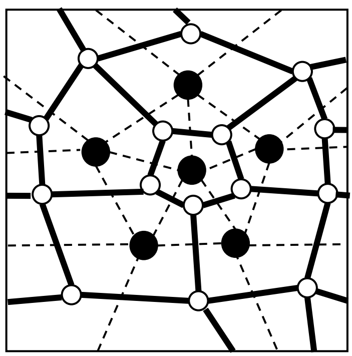
(ii)



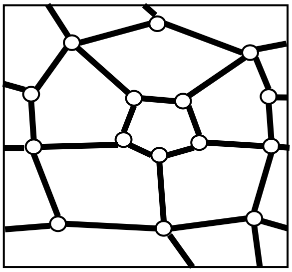
K_6 embedded in the KB

[The different types of lines are simply to make it easier to follow edges as they cross the identified edges with their reversed orientations.]

(iii) Joining a point in the middle of each face to the midpoint of each of the surrounding edges we get a map with 6 face s, each of which is adjacent to the other 5. This map requires 6 colours.



(iv) We need 6 colours to colour this map on the KB so $\mu(KB) \geq 6$.



(v) Heawood's formula gives

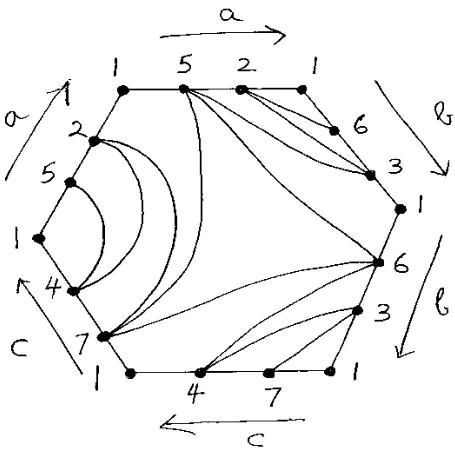
$$\mu(KB) \leq \text{INT}\left(\frac{7+\sqrt{24.2+1}}{2}\right) \text{ so } \mu(KB) \leq 7.$$

(vi) So $\mu(KB) = 6$ or 7.
(In fact it is 6 but we haven't shown that.)

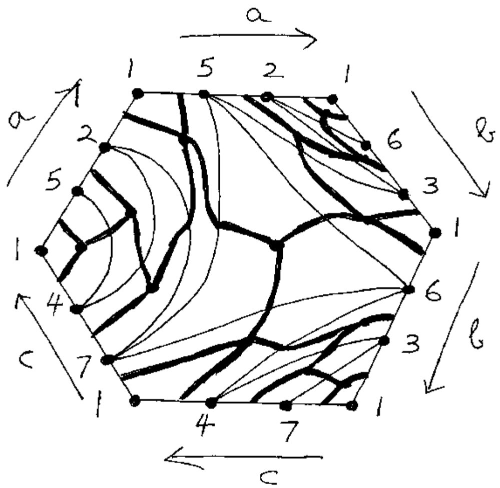
Exercise 7: (a) The weight of 3**P** is 3.

By Heawood's formula $n = \text{INT}\left(\frac{7+\sqrt{73}}{2}\right) = \mathbf{7}$.

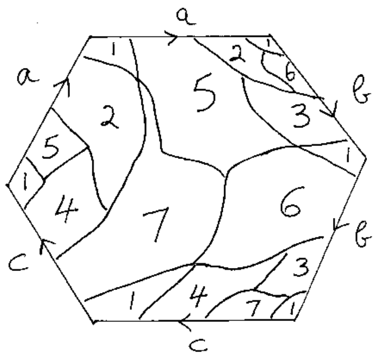
(b), (c)



(d) To obtain a map with 7 regions, each adjacent to the other 6, we take a point inside each face of the map obtained in (b) and join it to a point along each edge.



Removing the original edges we obtain a map on 3P in which each of the 7 regions are adjacent to the other 6. This map needs 7 colours.



Exercise 8:

(i) μ(Möbius Band) = μ(Projective Plane)

$$= \text{INT}\left[\frac{7 + \sqrt{24 + 1}}{2}\right] = 6.$$

(ii) μ(cylinder with 3 holes) = μ(sphere with 5 holes)

$= \mu$(sphere) = 4.

(iii) μ(sum of 10 projective planes)

$$= \text{INT}\left[\frac{7 + \sqrt{240 + 1}}{2}\right] = 11.$$

Exercise 9: (a) Let there be *p* pentagons and *h* hexagons. If V, F and E are the numbers of vertices, faces and edges respectively then:

$F = p + h$

$2E = 5p + 6h$ (each edge belongs to 2 faces)

$3V = 5p + 6h$ (each vertex belongs to 3 faces)

Since $V + F - E = 2$ we have

$$\frac{5p + 6h}{3} + p + h - \frac{5p + 6h}{2} = 2.$$

This reduces to $p = 12$.

(b) Clearly $F = 32$ and $h = 20$.

Hence $E = 90$ and $V = 60$.